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REMARKS ON MR. V. NARASIMHA MURTHI'S PAPER: ON A PROBLEM OF ARRANGEMENTS (I)

By F. W. LEVI, *Calcutta University.*

[Received 12 March 1940].

The problem of *arranging* $2n+1$ *persons on* n ***different*** *days around a circular table in such a way that no person has the same neighbour on different days*, can be considered also as a problem on graphs, and the main result published in the paper under consideration then reads as follows:

Every complete graph with $2n+1$ *vertices is a product of* n *simple circuits of* $2n+1$ *sides each.*

Of course, to every "dinner-party" there corresponds a simple circuit with the numbers $1, 2, \ldots, 2n+1$ as its vertices. If the n dinner-parties are arranged as required by the problem, the n circuits put together form a graph with $2n+1$ vertices where every pair of vertices is connected by exactly one segment. Conversely, every representation of a complete graph as a product of n simple circuits furnishes a solution of the problem of the dinner-parties. Two products of circuits are said to be *isomorphic* when the corresponding graphs can be put in $(1,1)$ correspondence such that the circuits of the first product correspond to the circuits of the second product. Starting from any solution of the dinner-party-problem, one gets new solutions: (a) by interchanging the figures $1, \ldots, 2n+1$, (b) by interchanging the n dinner-parties, and (c) by cyclic or anticyclic permutation inside the dinner-parties. Each of these alternations transforms the corresponding product of circuits into an isomorphic product, but only such solutions of the problem should be considered to be *different*, as correspond to non-isomorphic products of circuits. To state that two solutions which are given arithmetically are different in that sense, one must find out properties of the sets of circuits which are not properties of the other set, but which are invariant for every isomorphism.

For example, the three circuits

1 2 3 4 5 6 7; 1 3 5 7 2 4 6; 1 4 7 3 6 2 5

are connected in such a way that by overleaping a constant number of vertices in running over any one of the circuits one gets every other one. This property is invariant for every isomorphism. Thus, as the same regularity does not exist for

1 2 3 4 5 6 7; 1 3 5 2 7 4 6; 1 5 7 3 6 2 4,

this set of dinner-parties is different from the first one. Solutions of the first type exist for every prime number $2n+1$, whereas for $n > 2$, the solutions described in § 3 of the paper mentioned in the title, obviously have not that property. Hence for every prime number $p > 5$, there exist at least two different solutions of the problem. It would be interesting to know the number of different solutions for every value of n, as well as to get some information about the structure of these solutions (*e.g.* about the groups of automorphisms).

Dr. S. Chowla* and Mr. V. Narasimha Murti have mentioned a general solution of the "dinner-party-problem" which I have communicated privately to some friends. This result was obtained by a slight extension of a method used before by Dr. S. Chowla in a special case; it is not different from the solution in § 3 of Mr. Narasimha Murti's paper. Apparently this author got the same result by a different method and independently.

* *Proc. Ind. Acad. Sc* 9 (1939), 421.

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TYPES OF SOLUTIONS OF $x^3+y^3+z^3=1$* IN INTEGERS (I)

By M. VENKATARAMA AYYAR and M. BHIMASENA RAO,

Bangalore.

[Received 12 February 1940]

1. *Introduction.* S. Ramanujan proposed the following†

QUESTION: *Show that* $(6a^2-4ab+4b^2)^3=(3a^2+5ab-5b^2)^3+(4a^2-4ab+6b^2)^3+(5a^2-5ab-3b^2)^3$ *and find other quadratic expressions satisfying similar relations.* Later, he gave another‡

QUESTION: *Solve in integers* $x^3+y^3+z^3=1$ *and deduce*

$$6^3+8^3=9^3-1 \quad \text{(R. 1)}$$

$$9^3+10^3=12^3+1 \quad \text{(R. 2)}$$

$$135^3+138^3=172^3-1 \quad \text{(R. 3)}$$

$$791^3+812^3=1010^3-1 \quad \text{(R. 4)}$$

$$11161^3+11468^3=14258^3+1 \quad \text{(R. 5)}$$

$$65601^3+67402^3=83802^3+1. \quad \text{(R. 6)}$$

Prof. N. B. Mitra gave a partial solution§ taking $x=3a(1-3a^3)$, $y=9a^4$ and $z=1-9a^3$ and deduced the first two results.‡ Our object in this paper is to show the underlying unity in all the six examples of Ramanujan, reducing them and his algebraic identity to one type and to extend the work to several more types.

* Presented to the Eleventh Conference of the Indian Mathematical Society held at Hyderabad, December, 1939.

† *J. I. M. S.* (1) 5 (1913), p. 39, Question 441. Mr. S. Narayanan has given a particular solution of this in *J. I. M. S.* (1) 6 (1914), p. 226.

‡ *J. I. M. S.* (1) 7 (1915), p. 160, Question 681.

§ *J. I. M, S.* (1) 13 (1921), p. 17. There is another Question (661) by Ramanujan:—*Solve in integers* $x^3+y^3+z^3=u^6$, etc. solved by Prof. Mitra in *J. I. M. S.* (1) 13 (1921), p. 15.

2. *General Solution of* $x^3+y^3+z^3=1$.

Consider
$$x^3+y^3=z^3+w^3.$$

Put
$$x=L+\frac{LN-M^2}{3R},\quad y=L-\frac{LN-M^2}{3R}$$
$$z=M+\frac{MN-L^2}{3R},\quad w=M-\frac{MN-L^2}{3R},$$

then, provided that
$$LM=N^2+3R^2,$$

we get $$x^3+y^3=z^3+w^3=2LM[L^3-2LMN+M^3].$$

Now, $M/L=(z+w)/(x+y)=t$, say, which may be defined as the *Type* of the formula or example. Then, on putting

$$x=\frac{\lambda+\mu}{2},\quad y=\frac{\lambda-\mu}{2},\quad z=\lambda t-1,\quad w=1,$$

we get

$$\left(\frac{\lambda+\mu}{2}\right)^3+\left(\frac{\lambda-\mu}{2}\right)^3+(1-\lambda t)^3=1. \qquad (1)$$

3. *Convenient forms for special types.* It is seen that in $x^3+y^3=z^3+w^3$, $(x+y-z-w)$ is divisible by 2, and as

$$x^3+y^3-z^3-w^3$$
$$=(x+y-z-w)\{(x+y-z-w)^2+3(x+y)(z+w)\}$$
$$-3xy(x+y)+3zw(z+w),$$

$(x+y-z-w)$ is divisible by 3 and therefore, in all, by 6. But $(x+y-z-w)=\lambda(1-t)$. Hence, when $(t-1)$ is divisible by 2, λ should be divisible by 3; when $(t-1)$ is divisible by 3, λ should be divisible by 2; when $(t-1)$ is divisible by 6, λ may be quite arbitrary, and when $(t-1)$ is divisible neither by 2 nor by 3, λ should be divisible by 6. Also, as $\lambda^2 t=LM=N^2+3R^2$ each of λ and t should be of the form a^2+3b^2. Thus, for Type 3, with $a=0$, $b=1$, we can write 3λ for λ and get

$$\left(\frac{3\lambda+\mu}{2}\right)^3+\left(\frac{3\lambda-\mu}{2}\right)^3+(1-9\lambda)^3=1$$

i.e.
$$321\lambda^2-108\lambda+12=\mu^2. \qquad (2)$$

For Type 4, similarly, λ and μ being even,

$$(\lambda+\mu)^3+(\lambda-\mu)^3+(1-8\lambda)^3=1$$

i.e.
$$85\lambda^2-32\lambda+4=\mu^2. \qquad (3)$$

For Type 7, λ arbitrary,

$$\left(\frac{\lambda+\mu}{2}\right)^3+\left(\frac{\lambda-\mu}{2}\right)^3+(1-7\lambda)^3=1$$

i.e.
$$457\lambda^2-196\lambda+28=\mu^2. \qquad (4)$$

For Type 12, where λ is a multiple of 6, and μ even,

$$(3\lambda+\mu)^3+(3\lambda-\mu)^3+(1-72\lambda)^3=1$$

i.e.
$$20733\lambda^2-864\lambda+12=\mu^2. \qquad (5)$$

And so on, for other Types 13, 19, etc.

4. *General identity for $x^3+y^3=z^3+w^3$.*

In the initial set of values for x, y, z, w, put $M=Lt$ and remove the common factor $L/3R$. Then,

$$x=(3R+N)-Lt^2,\quad y=(3R-N)+Lt^2$$

$$z=(3R+N)t-L,\quad w=(3R-N)t+L.$$

Here $t=l^2+3m^2$, let $L=a^2+3b^2$. Then,

$$LM=(a^2+3b^2)^2(l^2+3m^2)$$
$$=[(a^2-3b^2)^2+3(2ab)^2](l^2+3m^2)$$
$$=N^2+3R^2,$$

where
$$N=(a^2-3b^2)l+6abm$$
and
$$R=(a^2-3b^2)m-2abl.$$

With these, we get

$$\Big[a^2\{(3m+l)-(l^2+3m^2)^2\}$$
$$-6ab(l-m)-3b^2\{(3m+l)+(l^2+3m^2)^2\}\Big]^3$$

$$+\Big[a^2\{(3m-l)+(l^2+3m^2)^2\}$$
$$-6ab(l+m)-3b^2\{(3m-l)-(l^2+3m^2)^2\}\Big]^3$$

$$=\Big[a^2\{(l^2+3m^2)(3m+l)-1\}$$
$$-6ab(l-m)(l^2+3m^2)-3b^2\{(l^2+3m^2)(3m+l)+1\}\Big]^3$$
$$+\Big[a^2\{(l^2+3m^2)(3m-l)+1\}$$
$$-6ab(l+m)(l^2+3m^2)-3b^2\{(l^2+3m^2)(3m-l)-1\}\Big]^3. \quad (6)$$

5. *Type* 4. First, let us consider Type 4 to which Ramanujan's identity and examples belong. Also, let us write the quadratic expression $La^2+2\,Mab+Nb^2$ in the form $(L,\ M,\ N)$. Then, on putting $l=2,\ m=0$ in (6), we get

$$(-14,-6,-54)^3+(14,-6,+54)^3=(7,-24,-27)^3+(-7,-24,+27)^3. \quad (7\text{-}a)$$

Or again, with $l=1,\ m=1$, we get, after removing the common factor 3 in each expression,

$$(-4,0,-20)^3+(6,-2,14)^3=(5,0,-17)^3+(3,-8,-7)^3. \quad (7\text{-}b)$$

In (7-*a*), write $3b$ for b, we get

$$(-14,-2,-6)^3+(14,-2,6)^3=(7,-8,-3)^3+(-7,-8,+3)^{3}\,{}^{*}. \quad (7\text{-}c)$$

In (7-*a*), changing a to $9b$, b to a and dividing by the common factor -27, we get

$$(2,2,42)^3+(-2,2,-42)^3=(1,8,-21)^3+(-1,8,21)^3, \quad (7\text{-}d)$$

where, on putting $a=d$ and $b=1$, we obtain a formula due to J. R. Young†.

In (7-*d*), replace a by $(3a+3b)$ and b by $(a-b)$, i.e. effect the transformation $\begin{pmatrix} 3 & 3 \\ 1 & -1 \end{pmatrix}$; on removing the common factor 12 from each expression, we get

* A formula given by Gerardin, in *L'Intermediare des Sciences Mathematiques*, 19 (1912), 7. *Vide* Dickson's *History of the Theory of Numbers*, Vol. 2., p. 559.

† Vide Dickson, *loc. cit.*, Vol, 2., p. 559.

It may be noticed here that (7-*d*) can be got directly thus:—

$(z+w)=t(x+y)$. Put $x+y=r,\ x-y=s,\ z+w=p$, $z-w=q,\ p=rt,\ p\,(p^2+3q^2)=r(r^2+3s^2)$;

hence $s^2-q^2t=r^2(t^3-1)/3$; $t=4$. Hence $s^2-21r^2=(2q)^2$.

If $s=a^2+21b^2,\ r=2ab,\ 2q=a^2-21b^2,\ p=8ab$,

hence $2x=r+s=a^2+2ab+21b^2$ etc.

$$(6a^2-4ab+4b^2)^3-(4a^2-4ab+6b^2)^3$$
$$=(3a^2+5ab-5b^2)^3+(5a^2-5ab-3b^2)^3$$

which is Ramanujan's identity.

In this Type 4, we have from (3) the equation

$$85\lambda^2-32\lambda+4=\mu^2$$

evidently satisfied by $\lambda=0$, $\mu=2$. Put the left side equal to

$\left(\lambda.\frac{p}{q}-2\right)^2$, then

$$\lambda=\frac{4q(p-8q)}{p^2-85q^2} \text{ and } \mu=\frac{4p(p-8q)}{p^2-85q^2}-2$$

Now, $$(85)^{\frac{1}{2}}=9+\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{18},$$
$$\quad * \qquad\qquad\qquad\qquad *$$

whose successive convergents p/q and corresponding rational divisors (R) occuring in the first period, are

p/q	1/0	9/1	37/4	46/9	83/9	378/41
R	1	−4	+9	−9	+4	−1.

Here, considering only such useful values of R as are factors of the determinant −84 of the quadratic form (85, −16, 4) and the corresponding p and q, we get

p	q	λ	μ	Result	
1	0	0	−2	$-2^3+2^3+1^3=1.$	(trivial) (8.0)
9	1	−1	−11	$-12^3+10^3+9^3=1.$	(8.1)
9	−1	17	−155	$-138^3+172^3-135^3=1.$	(8.2)
83	9	99	911	$1010^3-812^2-791^3=1.$	(8.3)

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p	q	λ	μ	Result
83	−9	−1395	12863	$11468^3 - 14258^3 + 11161^3 = 1.$ (8.4)
378	41	−8200	−75602	$-83802^3 + 67402^3 + 65601^3 = 1.$ (8.5)
378	−41	115784	−1067474	$-951690^3 + 1183258^3 - 926271^3 = 1.$ (8.6)

Now, (8.1)-(8.5) are (R.2)-(R. 6) of Ramanujan. (8.6) is extra. (R.1) is a mere combination of (8.1) or (R.2) with Euler's identity written as

$$6^3 + 8^3 = 12^3 - 10^3 = 9^3 - 1.$$

So far, we have given a method of getting numerical examples. But, we have yet to determine how many independent general solutions can be found for this type. The equation for Type 4 is

$$85\lambda^2 - 32\lambda + 4 = \mu^2,$$

i.e.

$$x^2 - 85y^2 = -84,$$

where

$$x = 85\lambda - 16, \ y = \mu.$$

We have first to solve $N^2 \equiv 85 \pmod{-84}$, whose roots are $N = \pm 1, \pm 13, \pm 29, \pm 41$. Now, the reduced forms belonging to the principal class of forms equivalent to $(1, 0, -85)$ are as below.

$$(1, 9, -4) = \begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix} \quad (1, 0, -85)$$

$$(-4, 7, 9) = \begin{pmatrix} -9 & 37 \\ -1 & 4 \end{pmatrix} \quad ,,$$

$$(9, 2, -9) = \begin{pmatrix} -37 & -46 \\ -4 & -5 \end{pmatrix} \quad ,,$$

$$(-9, 7, 4) = \begin{pmatrix} 46 & -83 \\ 5 & -9 \end{pmatrix} \quad ,,$$

$$(4, 9, -1) = \begin{pmatrix} 83 & 378 \\ 9 & 41 \end{pmatrix} \quad ,,$$

$$(-1, 9, 4) = \begin{pmatrix} -378 & 6887 \\ -41 & 747 \end{pmatrix} \quad ,,$$

$$(4, 7, -9) = \begin{pmatrix} -6887 & -27926 \\ -747 & -3029 \end{pmatrix} (1, 0, -85)$$

$$(-9, 2, 9) = \begin{pmatrix} 27926 & -34813 \\ 3029 & -3776 \end{pmatrix} \quad ,,$$

$$(9, 7, -4) = \begin{pmatrix} 34813 & 62739 \\ 3776 & 6805 \end{pmatrix} \quad ,,$$

$$(-4, 9, 1) = \begin{pmatrix} -62739 & 285769 \\ -6805 & 30996 \end{pmatrix} \quad ,,$$

Now, from the roots of the congruence, with $M=-84$, $L=(N^2-D)/M$,

N	± 1	± 13	± 29	± 41
L	1	-1	-9	-19

we get eight forms (M, N, L).

O. hese, the first $(-84, 1, 1)$ becomes $(1, 9, -4)$ on using the transf rmation $\begin{pmatrix} 0 & 1 \\ -1 & 10 \end{pmatrix}$ and the latter occurs in the principal class bove. Hence

$$(-84, 1, 1) = \begin{pmatrix} -10 & -1 \\ 1 & 0 \end{pmatrix} (1, 9, -4)$$

$$= \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} (1, 0, -85).$$

Hence $x^2-85y^2=-84$ is satisfied by $x=-1$, $y=\pm 1$. But $x=85\lambda-16$; this value of x renders λ fractional. Similarly $(-84, -1, 1)$. Next,

$$(-84, 13, -1) \sim (-1, 9, 4), \text{ hence}$$

$$(-84, 13, -1) = \begin{pmatrix} -1429 & 378 \\ -155 & 41 \end{pmatrix} (1, 0, -85).$$

Hence $x^2-85y^2=-84$ is satisfied by $x=1429$, $y=155$.

But $85\lambda-16=1429$. Hence $\lambda=17$ and $\mu=\pm 155$.

If t and u are so chosen that $t^2-85u^2=1$, the general values of x and y are given by

$$x = 1429t + 85.155u$$

$$y = 1429u + 155t,$$

so that

$$\left.\begin{aligned} \lambda &= 155u+17t-16.\frac{t-1}{85} \\ \mu &= 1429u+155t, \\ \text{where}\quad 1 &= t^2-85u^2. \end{aligned}\right\} \tag{9.1}$$

Here $t=1$, $u=0$ gives $\lambda=17$, $\mu=155$, yielding (8.2).

If we take $(-84, -13, -1)=\begin{pmatrix} 8399 & 378 \\ 911 & 41 \end{pmatrix}(1, 0, -85)$

we get $x=8399$, $y=911$ giving as general values

$$x=8399t+85.911u$$

$$y=\ 911t+8399u$$

with $$1=t^2-85u^2,$$

but this set is not different from (9.1)*, for on putting

$$t=285769 \text{ and } u=-30996$$

we get $$x=-1429 \text{ and } y=155 \text{ as in } (9.1).$$

Thus, in (9.1), $t=-285769$ and $u=30996$ gives $\lambda=99$, $\mu=911$, and we obtain (8.3). Next,

$$(-84, 29, -9)=\begin{pmatrix} 101 & -46 \\ 11 & 5 \end{pmatrix}(1, 0, -85).$$

Hence, proceeding as before, $x=101t+85.11u$

$$y=101u+11t$$

and hence

$$\left.\begin{aligned} \lambda &= 11u+t+16\frac{t+1}{85} \\ \mu &= 101u+11t \\ \text{with}\quad 1 &= t^2-85u^2. \end{aligned}\right\} \tag{9.2}$$

Here, $t=285769$, $u=-30996$ give λ and μ as in (8.4). $N=\pm 41$ leads to fractional λ.

The above two solutions are those based on primitive representations. If we go to the derived ones, put $x=2x'$, $y=2y'$, $x'^2-85y'^2=-21$ with $N^2\equiv 85 \pmod{-21}$ whose roots are

N	± 1	± 8
L	4	1.

* It can be proved that $(M, \pm N, L)$ both lead to the same set of solutions.

Now, $(-21, \pm 1, 4)$ do not lead to integral λ. But

$$(-21, 8, 1) = \begin{pmatrix} -17 & -1 \\ 1 & 0 \end{pmatrix} (1, 9, -4)$$

$$= \begin{pmatrix} -8 & -1 \\ 1 & 0 \end{pmatrix} (1, 0, -85).$$

Hence $x' = \pm 8$, $y' = \pm 1$ is a solution of $x'^2 - 85y'^2 = -21$.

Hence $x = 16$, $y = 2$ is a solution of $x^2 - 85y^2 = -84$.

The general values are $x = 16t + 170u$

$y = 16u + 2t$

giving

$$\left.\begin{aligned} \lambda &= 2u + \frac{16(1+t)}{85} \\ \mu &= 16u + 2t \\ 1 &= t^2 - 85u^2. \end{aligned}\right\} \qquad (9.3)$$

with

Here, $t = -1$, $u = 0$ leads to the trivial (8.0),

$t = 285769$, $u = 30996$ yields (8.5)

and $t = 285769$, $u = -30996$ yields (8.6).

Thus, in Type 4, all the examples fall under three sets, two based on primitive representation and one on derived representation.

Now, we shall obtain the identities corresponding to these three sets. Starting with (*7-d*) viz.

$$(2, 2, 42)^3 + (-2, 2, -42)^3 = (1, 8, -21)^3 + (-1, 8, 21)^3,$$

make the transformation $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We get

$$(2, 4, 48)^3 + (-2, 0, -20)^3 = (\mathit{1, 9, -4})^3 + (1, 7, 36)^3. \qquad (10.1)$$

Here $(1, 9, -4)$ comes under the principal class and can be made unity. Again, apply to (*7-d*) the transformation $\begin{pmatrix} 1 & -4 \\ 1 & 0 \end{pmatrix}$ and remove the common factor -4. We get

$$(-12, 4, -8)^3 + (10, 0, 8)^3 = (\mathit{1, 9, -4})^3 + (\mathit{-9, 7, 4})^3.$$

The forms in italics here may either of them be made unity and each will give rise to a distinct set. Thus, there are in all three distinct sets of identities in Type 4. As illustrations, put $b = 0$ in (10.1), we get $-12^3 + 10^3 + 9^3 = 1$ which is (8.1); put

$a=9$, $b=5$ in (10.2), we get $-812^3+1010^3-791^3=1$ which is (8.3), and so on.

6. *Type* 3.—The equation of this type, viz.

$$321\lambda^2-108\lambda+12=\mu^2$$

is satisfied by $\lambda=1$, $\mu=15$. Put $\lambda=\rho+1$ and equate the left side to $\left(\rho\frac{p}{q}+15\right)^2$. Then,

$$\rho=\frac{6q(89q-5p)}{p^2-321q^2},\ \lambda=\rho+1$$

and
$$\mu=\frac{6p(89q-5p)}{p^2-321q^2}+15.$$

Now, $(321)^{\frac{1}{2}}=17+\frac{1}{1}+\frac{1}{10}+\frac{1}{1}+\frac{1}{34}$ with
* *

successive $\frac{p}{q}$	$\frac{1}{0}$	$\frac{17}{1}$	$\frac{18}{1}$	$\frac{197}{11}$	$\frac{215}{12}$
corresponding R	1	-32	3	-32	1.

Rejecting the rational divisors that are not factors of -936, the determinant in the transformed equation, we obtain

p	q	λ	μ	Result	
1	0	1	-15	$-6^3+9^3-8^3=1.$	(11.1)
18	1	-1	-21	$-12^3+9^3+10^3=1.$	(11.2)
18	-1	359	-6429	$-2676^3+3753^3-3230^3=1.$	(11.3)
215	12	-503	-9015	$-5262^3+3753^3+4528^3=1.$	(11.4)
215	-12	154297	-2764455	$-1150782^3+1613673^3-1388672^3=1.$	(11.5)

Here, it should be noticed that (11.1) and (11.2) as well as (11.3) and (11.4) could be written combined thus:—

$$9^3-1=8^3+6^3=12^3-10^3 \qquad (11.a)$$

$$3753^3-1=5262^3-4528^3=3230^3+2676^3. \qquad (11.b)$$

In $321\lambda^2-108\lambda+12=\mu^2$, put $\mu=3y$, remove 3, write $107\lambda-18=x$, $\mu=y$. Then $x^2-321y^2=-104$, the corresponding congruence being $N^2\equiv 321 \pmod{-104}$. The roots of this are

N	± 3	± 23	± 29	± 49
L	3	-2	-5	-20

with $M=-104$.

Belonging to the principal class of $(1, 0, -321)$, we have

$$(1, 17, -32) = \begin{pmatrix} 1 & 17 \\ 0 & 1 \end{pmatrix} \quad (1, 0, -321)$$

$$(-32, 15, 3) = \begin{pmatrix} -17 & 18 \\ -1 & 1 \end{pmatrix} \quad ,,$$

$$(3, 15, -32) = \begin{pmatrix} -18- & 197 \\ -1- & 11 \end{pmatrix} \quad ,,$$

$$(-32, 17, 1) = \begin{pmatrix} 197 & -215 \\ 11 & -12 \end{pmatrix} \quad ,,$$

Now,

$$(-104, 3, 3) = \begin{pmatrix} -6 & -1 \\ 1 & 0 \end{pmatrix} \quad (3, 15, -32)$$

$$= \begin{pmatrix} -89 & 18 \\ -5 & 1 \end{pmatrix} \quad (1, 0, -321).$$

Hence $x=89$, $y=5$ is a solution of

$$x^2-321y^2=-104.$$

Hence the general value (on combining $89^2-321.5^2=-104$ with $t^2-321u^2=1$)

is

$$x=89t+321.5u$$

$$y=89u+5t,$$

and

$$\left.\begin{aligned} \lambda &= t+15u-\frac{18(t-1)}{107} \\ \mu &= 3(5t+89u), \\ 1 &= t^2-321u^2. \end{aligned}\right\} \qquad (12)$$

where

Here $t=1$, $u=0$ yields (11.1)

$t=215$, $u=-12$ yields (11.2)

$t=215$, $u=+12$ yields (11.3), and so on.

The same result can be got from $(-104, -3, 3)$, while from $(-104, \pm 29, -5)$ we get a result which can be derived from the former. Thus (12) includes all the formulæ that can be got from primitive representation. The derived representation with $N^2 \equiv 321 \pmod{-26}$ whose roots are $N = \pm 3$ and therefore for $M=-26$, $L=12$ gives rise to the forms $(-26, \pm 3, 12)$ which are improperly primitive and are not useful as they cannot be obtained by unitary transformations from the properly primitive $(1, 0, -321)$. Thus in Type 3, (12) is the one fundamental result from which could be derived all possible numerical results.

In (6), put $\lambda = 0$, $\mu = 1$ and remove the common factor 2. We get

$$(-3a^2+3ab-18b^2)^3+(6a^2-3ab+9b^2)^3$$
$$=(4a^2+9ab-15b^2)^3+(5a^2-9ab-12b^2)^3.$$

Apply the transformation $\begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}$, remove -2, put $2b$ for b; then we get

$$(9, -3, 24)^3+(-6, 9, -48)^3+(-8, -1, +40)^3=(1, 17, -32)^3. \tag{13}$$

Writing this as

$$(9, -3, 24)^3-(1, 17, -32)^3=(6, -9, +48)^3+(8, 1, -40)^3,$$

if we transform by $\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$ and remove the common factor 3, we get the left side unaltered and different expressions on the right side. Thus

$$\left.\begin{aligned}(9, -3, 24)^3-(1, 17, -32)^3 &= (6, -9, 48)^3+(8, 1, -40)^3\\ &= (12, -15, 36)^3+(-10, 19, -4)^3.\end{aligned}\right\} \tag{14}$$

Here $(1, 17, -32)$ can be made unity. This double equality comes out in Types 3, $3k^2$, 7, 13, 19 but is not found in Type 4. The reason for this is explained later.

7. *Type* $3k^2$.—In (1), put $t=3k^2$. We have

$$(108k^6-1)\lambda^2-108k^4\lambda+36k^2 = 3\mu^2.$$

Changing λ to 3λ, we get

$$3(108k^6-1)\lambda^2-108k^4\lambda+12k^2 = \mu^2.$$

When $\lambda = k$, $\mu = 18k^4-3k$. Put $\lambda = \rho+k$ and write the left side $=\left[\rho\frac{p}{q}+(18k^4-3k)\right]^2$. Then,

$$\rho = \frac{6kq[q(108k^6-18k^3-1)-p(6k^3-1)]}{p^2-3(108k^6-1)q^2}.$$

Put $3k^3 = s$. Then $3(108k^6-1) = 36s^2-3$.

Now $$(36s^2-3)^{\frac{1}{2}} = (6s-1)+\frac{1}{\overset{}{1}+}\ \frac{1}{(4s-2)+}\ \frac{1}{1+}\ \frac{1}{(12s-2)}.$$
$\quad\quad\quad\quad\quad\quad\quad\quad$ * $\quad\quad\quad\quad\quad\quad\quad\quad$ *

Here,

successive $\frac{p}{q}=$	$\frac{1}{0}$	$\frac{6s-1}{1}$		$\frac{6s}{1}$	$\frac{24s^2-6s-1}{4s-1}$	$\frac{24s^2-1}{4s}$
corresponding $R=$	1	$-(12s-4)$		3	$-(12s-4)$	1,

whence we obtain

p	q	Result	
1	0	$(9k^4)^3-(9k^4+3k)^3+(9k^3+1)^3 = 1$*.	(15.1)
$6s$	-1	$(-3888k^{10}-1296k^7-81k^4+3k)^3+(3888k^{10}-135k^4)^3 + (3888k^9+648k^6-9k^3+1)^3 = 1.$	(15.2)
$24s^2-1$	$4s$	$-9s(2304s^5-2304s^4+336s^3+144s^2-7s-1)^3 - (6912s^5-3456s^4-432s^3+72s^2+3s-1)^3 + 9s(2304s^5-816s^3+17s)^3 = 1.$	(15.3)

Here, for Type 12, $k=\pm 2$, $s=\pm 24$ and we get from (15.1) and (15.2),

$$150^3-73^3 = 138^3+71^3 = 144^3-1 \qquad (16.1)$$

* Given by Gerardin in *L'Intermediare des Sc. Math.* 19 (1912), 7. *Vide* Dickson *loc. cit.*

$$4148490^3 - 2032057^3 = 3816726^3 + 1949111^3$$
$$= 3979152^3 - 1. \qquad (16.2)$$

For Type 27, $k = \pm 3$, $s = \pm 81$ and (15.1) gives

$$738^3 - 244^3 = 720^3 + 242^3 = 729^3 - 1 \qquad (17)$$

and so on.

For this type, put $l = 0$, $m = k$ in (6). Change a to $(a-b)$ and b to $(a+b)$, remove 4, then if

$$X = a + ab + b^2$$
$$Y = a^2 + 2ab$$
$$Z = 2ab + b^2,$$

$$(X + 9k^3Z)^3 + (-X + 9k^3Y)^3$$
$$= (9k^4X + 3kZ)^3 + (-9k^4X + 3kY)^3. \qquad (18.a)$$

Here make the transformation $\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$, cancel 3, the first term on each side is unaltered; if $W = a^2 - b^2$, we get the double equality $(X + 9k^3Z)^3 - (9k^4X + 3kZ)^3$

$$= (X - 9k^3Y)^3 - (9k^4X - 3kY)^3$$
$$= (X + 9k^3W)^3 - (9k^4X + 3kW)^3. \qquad (18)$$

8. *Type 7.* Here, from (4), the formula is

$$\left(\frac{\lambda+\mu}{2}\right)^3 + \left(\frac{\lambda-\mu}{2}\right)^3 + (1-7\lambda)^3 = 1$$

i.e. $$457\lambda^2 - 196\lambda + 28 = \mu^2.$$

Evidently (or from $9^3 - 8^3 - 6^3 = 1$), $\lambda = 1$ gives $\mu = 17$.

Put $\lambda = \rho + 1$ and left side $= (\rho p/q + 17)^2$.

Then $$\rho = \frac{2q(359q - 17p)}{p^2 - 457q^2}$$

with $\lambda = \rho + 1$ and $\mu = \rho \dfrac{p}{q} + 17$.

Now, $(457)^{\frac{1}{2}} = 21 + \dfrac{1}{2+}\,\dfrac{1}{1+}\,\dfrac{1}{1+}\,\dfrac{1}{1+}\,\dfrac{1}{5+}\,\dfrac{1}{2}\cdots\dfrac{1}{+42}$.
$\quad$ * $\qquad\qquad\qquad\qquad$ *

Here using such values of p and q as correspond to those rational divisors which are factors of the determinant -3192, we have

p	q	λ	μ	Result
1	0	1	17	$-8^3+9^3-6^3=1$. (19.1)
43	-2	277	-5917	$-2820^3+3097^3-1938^3=1$. (19.2)
107	$+5$	-9	-197	$-103^3+94^3+64^3=1$. (19.3)
171	$+8$	81	1727	$904^3-823^3-566^3=1$. (19.4)
962	-45	153991	-3291947	$-1568978^3+1722969^3-1077936^5=1$. (19.5)
2095	-98	-4625403	98879827	$47127212^3-51752615^3+32377822^3=1$. (19.6)
149365	6987	53925667	1152799087	$603362377^3-549436710^3-377479668^3=1$. (19.7)

If, at the start, instead of $\lambda=1$ and $\mu=17$ we start with $\lambda=-9$ and $\mu=197$ and then put $\lambda=\rho-9$, by proceeding as before, we get

$$\lambda=\frac{-2q(4211q+197p)}{p^2-457q^2}-9$$

$$\mu=\frac{p}{q}\left[\frac{-2q(4211q+197p)}{p^2-457q^2}\right]+197.$$

Here, for

p	q	λ	μ	Result
865898	−40505	−59218319	1265943073	$603862377^3 - 662580696^3 + 414528234^3 = 1.$

(19.8)

Now, if $x = 457\lambda - 98$ and $y = \mu$, the equation of this type is $x^2 - 457y^2 = -3192$, the corresponding congruence equation therefore being

$$N^2 \equiv 457 \pmod{-3192}$$

whose roots are

$N =$	115	151	227	305	493	571	647	683
$L =$	−4	−7	−16	−29	−76	−102	−131	−146
$N =$	913	949	1025	1103	1291	1369	1445	1481
$L =$	−261	−282	−329	−381	−522	−587	−654	−687.

Now the quadratic forms belonging to the principal class of forms equivalent to $(1, 0, -457)$ are

$$(1, 21, -16) = \begin{pmatrix} 1 & 21 \\ 0 & 1 \end{pmatrix} \quad (1, 0, -457)$$

$$(-16, 11, 21) = \begin{pmatrix} -21 & 43 \\ -1 & 2 \end{pmatrix} \quad ,,$$

$$(21, 10, -17) = \begin{pmatrix} -43 & -64 \\ -2 & -3 \end{pmatrix} \quad ,,$$

$$(-17, 7, 24) = \begin{pmatrix} 64 & -107 \\ 3 & -5 \end{pmatrix} \quad ,,$$

$$(24, 17, -7) = \begin{pmatrix} 107 & 171 \\ 5 & 8 \end{pmatrix} \quad ,,$$

$$(-7, 18, 19) = \begin{pmatrix} -171 & 962 \\ -8 & 45 \end{pmatrix} \quad ,,$$

$$(19, 20, -3) = \begin{pmatrix} -962 & -2095 \\ -45 & -98 \end{pmatrix} \quad ,,$$

$$(-3, 19, 32) = \begin{pmatrix} 2095 & -28197 \\ 98 & -1319 \end{pmatrix} \quad (1, 0, -457)$$

$$(32, 13, -9) = \begin{pmatrix} 28197 & 30292 \\ 1319 & 1417 \end{pmatrix} \quad ,,$$

$$(-9, 14, 29) = \begin{pmatrix} -30292 & 119073 \\ -1417 & 5570 \end{pmatrix} \quad ,,$$

$$(29, 15, -8) = \begin{pmatrix} -119073 & -149365 \\ -5570 & -6987 \end{pmatrix} \quad ,,$$

$$(-8, 17, 21) = \begin{pmatrix} 149365 & -716533 \\ 6987 & -33518 \end{pmatrix} \quad ,,$$

$$(21, 4, -21) = \begin{pmatrix} 716533 & 865898 \\ 33518 & 40505 \end{pmatrix} \quad ,,$$

etc.

Now, $(-3192, \pm 115, -4)$ is improperly primitive.

Next, $(-3192, 151, -7) \sim (-7, 17, 24)$ a reduced form which occurs in the fourth quarter of the principal class of forms given above; but

$$(-3192, -151, -7) = \begin{pmatrix} 19 & 1 \\ 1 & 0 \end{pmatrix} \quad (-7, 18, 19)$$

$$= \begin{pmatrix} 4211 & 171 \\ 197 & 8 \end{pmatrix} \quad (1, 0, -457).$$

Hence $x = 4211$, $y = 197$ is a solution of $x^2 - 457y^2 = -3192$. If $t^2 - 457u^2 = 1$, the general solution is given by

$$x = 4211t + 457.197u$$

$$y = 197t + \quad 4211u$$

so that

$$\left.\begin{aligned} \lambda &= 197u + 9t + 98.\frac{t+1}{457} \\ \mu &= 197t + 4211u. \end{aligned}\right\} \tag{20.1}$$

Here $t = -1$, $u = 0$ gives $\lambda = -9$, $\mu = -197$ yielding (19.3).

$(-3192, \pm 227, -16)$ is improperly primitive.

$$(-3192, 305, -29) = \begin{pmatrix} 11 & -1 \\ 1 & 0 \end{pmatrix} \quad (-29, 14, 9)$$

$$= \begin{pmatrix} 70373789 & \cdots \\ 3291947 & \cdots \end{pmatrix} \quad (1, 0, -457).$$

Here $x = 70373789$, $y = 3291947$ is a solution of $x^2 - 457y^2 = -3192$. From the general values, we get

$$\left.\begin{aligned} \lambda &= 153991t+3291947u-98\,\frac{t-1}{457} \\ \mu &= 3291947t+70373789u, \\ \text{where} \quad 1 &= t^2-457u^2. \end{aligned}\right\} \tag{20.2}$$

Here $t=1$, $u=0$ leads to (19.5).

Also, $(-3192, -305, -29)$ will only lead to a form derivable from the above. Next,

$$(-3192, -647, -131) = \begin{pmatrix} 19 & 4 \\ -5 & 1 \end{pmatrix} (3, 20, -19)$$

$$= \begin{pmatrix} 2113809269 & \cdots \\ 98879827 & \cdots \end{pmatrix} (1, 0, -457).$$

Hence $x = 2113809269$, $y = 98879827$ is a solution which, through general values, leads to

$$\left.\begin{aligned} \lambda &= 4625403t+98879827u+98\,\frac{t+1}{457} \\ \mu &= 98879827t+2113809269u, \\ \text{where} \quad 1 &= t^2-457u^2. \end{aligned}\right\} \tag{20.3}$$

Here $t=-1$, $u=0$ leads to (19.6).

$(-3192, +647, -131)$ gives rise to no independent solution.

$(-3192, \pm 683, -146)$ is improperly primitive. Next,

$$(-3192, 913, -261) = \begin{pmatrix} 18 & -5 \\ -7 & 2 \end{pmatrix} (7, 17, -24)$$

$$= \begin{pmatrix} -24644029721 & \cdots \\ -1152799087 & \cdots \end{pmatrix} (1, 0, -457).$$

With these big numbers as values of x and y, using the general solution, we obtain

$$\left.\begin{aligned} \lambda &= 53925667t+1152799087u-98\,\frac{t-1}{457} \\ \mu &= 1152799087t+24644029721u, \\ \text{where} \quad 1 &= t^2-457u^2. \end{aligned}\right\} \tag{20.4}$$

Here $t=1$, $u=0$ leads to (19.7).

$(-3192, -913, -261)$ leads to the same form.

$(-3192, \pm 949, -282)$ is improperly primitive. Then,

$$(-3192, -1025, -329) = \begin{pmatrix} -3 & -1 \\ 19 & 6 \end{pmatrix} (19, 20, -3)$$

$$= \begin{pmatrix} -36919 & \cdots \\ -1727 & \cdots \end{pmatrix} (1, 0, -457).$$

These numbers as values of x and y give generally

$$\left.\begin{aligned} \lambda &= 1727u+81t-98\frac{t-1}{457} \\ \mu &= 1727t+36919u, \\ \text{where}\qquad 1 &= t^2-457u^2. \end{aligned}\right\} \tag{20.5}$$

Here $t=1$, $u=0$ gives $\lambda=81$, $\mu=1727$ yielding (19.4). $(-3192, 1025, -329)$ gives rise to a derivable solution. Next,

$$(-3192, -1103, -381) = \begin{pmatrix} 20 & 7 \\ -3 & -1 \end{pmatrix}(-3, 19, 32)$$

$$= \begin{pmatrix} 126491 & \cdots \\ 5917 & \cdots \end{pmatrix}(1, 0, -457).$$

With these numbers as values of x and y, we are led to

$$\left.\begin{aligned} \lambda &= 277t+5917u-98\frac{t-1}{457} \\ \mu &= 5917t+126491u, \\ \text{where}\qquad 1 &= t^2-457u^2. \end{aligned}\right\} \tag{20.6}$$

Here $t=1$, $u=0$ gives $\lambda=277$, $\mu=5917$ yielding (19.2). $(-3192, 1103, -381)$ gives rise only to a solution derivable from the above. Then,

$$(-3192, -1369, -587) = \begin{pmatrix} 16 & 7 \\ -7 & -3 \end{pmatrix}(7, 17, -24)$$

$$= \begin{pmatrix} -27062771881 & \cdots \\ -1265943073 & \cdots \end{pmatrix}(1, 0, -457).$$

These numbers as values of x and y lead to

$$\left.\begin{aligned} \lambda &= 59218319t+1265943073u+98\frac{t+1}{457} \\ \mu &= 1265943073t+27062771881u, \\ \text{where}\qquad 1 &= t^2-457u^2. \end{aligned}\right\} \tag{20.7}$$

Here $t=-1$, $u=0$ give λ and μ that yield (19.8). $(-3192, +1369, -587)$ leads to a dependent solution. $(-3192 \pm 1445, -654)$ is improperly primitive.

$$(-3192 + 1481, -687) = \begin{pmatrix} -13 & 6 \\ 2 & -1 \end{pmatrix}(-16, 11, 21)$$

$$= \begin{pmatrix} 359 & \cdots \\ 17 & \cdots \end{pmatrix}(1, 0, -457).$$

Hence $x=359$, $y=17$ is a solution of $x^2-457y^2=1$, and general values of x and y lead to

$$\left.\begin{aligned} \lambda &= t+17u-98\frac{t-1}{457} \\ \mu &= 17t+359u, \\ \text{where}\qquad 1 &= t^2-457u^2. \end{aligned}\right\} \tag{20.8}$$

Here $t = 1, u = 0$ gives $\lambda = 1, \mu = 17$ yielding (19.1). And $(-3192, -1481 -687)$ yields only a result that could be got from the above. Now, for Type 7,

$$\lambda = 2q.\frac{-17p+359q}{p^2-457q^2}+1$$

$$\mu = 2p.\frac{-17p+359q}{p^2-457q^2}+17.$$

If $(359q-17p)$ is divisible by (p^2-457q^2), we should have

$$359q-17p \equiv 0 \pmod{\theta}$$

and

$$p^2-457q^2 \equiv 0 \pmod{\theta}.$$

The two equations are satisfied if

$$359q^2-457.17^2=-3192 \equiv 0 \pmod{\theta}.$$

As $3192=2^3\times3\times7\times19$, the rational divisors, in the development of $\sqrt{457}$ as a C. F., which are factors of 3192 are probable values of θ. They are, in order, 1, 21, 24, 7, 19, 3, 8, 21, eight in all.

For $\theta = 1$, use the transformation $\begin{pmatrix} 1 & 21 \\ 0 & 1 \end{pmatrix}$, where 21 is the middle coefficient of the reduced form which begins with 1, i.e. put, $p = a+21b$ and $q = b$. We get

$$\lambda = \frac{a^2+8ab-12b^2}{a^2+42ab-16b^2}$$

and

$$\mu = \frac{-17a^2+4ab-188b^2}{a^2+42ab-16b^2}.$$

If

$$x = a^2+42ab-16b^2,$$

$$y = 7(a^2+8ab-12b^2)-x,$$

$$z = \frac{\lambda+\mu}{2}x$$

$$w = \frac{\lambda-\mu}{2}x$$

we have

$$x^3+y^3 = z^3+w^3.$$

$\therefore$ $(1, 21, -16)^3+(6, 7, -68)^3=(-8, 3, -100)^3+(9, 1, 88)^3$. (21.1)

Here $a = 1, b = 0$ gives (19.1).

Next, for $\theta = 21$, start with the above result and apply to it the transformation $\begin{pmatrix} \theta & h-\theta' \\ 0 & 1 \end{pmatrix}$ where h is the middle coefficient of

the reduced form whose first term is θ, and $\theta' = 21$. That is, in this case, applying $\begin{pmatrix} 21 & -11 \\ 0 & 1 \end{pmatrix}$ to the forms in (21.1), we get

$$(21, 10, -17)^3 + (126, -59, 24)^3 = (-168, 91, -54)^3 + (189, -98, 55)^3. \qquad (21.2)$$

Here $a = -3$, $b = 2$ yields (19.2).

Then, for $\theta = 24$ with $h = -17$, $h - \theta' = -38$, apply $\begin{pmatrix} 24 & -38 \\ 0 & 1 \end{pmatrix}$ to the forms in the result (21.1). On changing the sign of b, we get

$$(24, 17, -7)^3 + (144, 221, 336)^3 = (-192, -307, -495)^3 + (216, 341, 542)^3. \qquad (21.3)$$

Here $a = 8$, $b = -5$ yields (19.3).

Then, for $\theta = -7$, $h = -18$, $h - \theta' = -39$; apply $\begin{pmatrix} -7 & -39 \\ 0 & 1 \end{pmatrix}$ to (21.1) ; we get, on changing the sign of b,

$$(-7, 18, 19)^3 + (-42, 227, -1216)^3 = (56, -315, 1786)^3 + (-63, 350, -1957)^3. \qquad (21.4)$$

Here $a = 45$, $b = 8$ yields (19.4).

Then, for $\theta = 19$, $h = 20$, $h - \theta' = -1$, and apply $\begin{pmatrix} 19 & -1 \\ 0 & 1 \end{pmatrix}$ to (21.1) ; we get

$$(19, 20, -3)^3 + (114, 1, -4)^3 = (-152, 11, -6)^3 + (171, -8, 5)^3. \qquad (21.5)$$

Here $a = -98$, $b = 45$ yields (19.5).

Similarly for $\theta = -3$, apply $\begin{pmatrix} -3 & -2 \\ 0 & 1 \end{pmatrix}$ to (21.1) ; we get

$$(-3, 19, 32)^3 + (-18, -5, 24)^3 = (24, 19, 48)^3 + (-27, -19, -40)^3. \qquad (21.6)$$

Here, $a = -1319$, $b = -98$ yields (19.6).

Again, for $\theta = -8$, apply $\begin{pmatrix} -8 & -38 \\ 0 & 1 \end{pmatrix}$ to (21.1) ; change the sign of b, we have

$$(-8, 17, 21)^3 + (-48, 221, -1008)^3 = (64, -307, 1485)^3 + (-72, 341, -1626)^3. \qquad (21.7)$$

Starting with (21.1) however, one is not led to the result which would correspond to the repeated $\theta = 21$. Instead, apply $\begin{pmatrix} 7 & 6 \\ 0 & 1 \end{pmatrix}$ to the result (21.7), the first of the forms becomes $(-56, -31, -9)$,

but this is not a reduced form. Reduce this and therefore apply the combined transformation $\begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix}$ to all the forms in (21.7). Then we get

$$(32, 13, -9)^3 + (-116, 43, -12)^3 = (200, -47, 15)^3 + (-212, 55, -18)^3. \quad (21.8)$$

Here $a = 1417$ and $b = 1319$ yields (19.8).

Thus, we have, in all, eight distinct identities in Type 7.

9. *The occurrence of double equalities.* Before going to further types, let us enquire when we will get double equalities of the nature met with in Types 3, $3k^2$, etc. Now,

$$(z+w) = t(x+y)$$

$$z^3 + [t(x+y) - z]^3 = x^3 + y^3.$$

Hence $\quad t^3(x+y)^2 - 3t^2 z(x+y) + 3tz^2 = x^2 - xy + y^2.$

Hence

$$y^2(t^3-1) + y[3t^2(xt-z) - x(t^3-1)] + (t^3x - 3t^2zx + 3tz^2 - x^2) = 0.$$

If y and y_1 are the roots of this equation,

$$y + y_1 = x + \frac{3t^2(z-xt)}{t^3-1}.$$

Hence $\quad y_1 = (x-y) + \frac{3t^2(z-xt)}{t^3-1}.$

Then $\quad w_1 = (z-w) + 3\frac{z-xt}{t^3-1}.$

If y_1 and w_1 are integers, we get a second equality,

$$z^3 - x^3 = y^3 - w^3 = y_1^3 - w_1^3 *.$$

Again, as $\quad z^3 + w^3 = x^3 + y^3$

and $\quad z + w = t(x+y),$

$$t(z^2 - zw + w^2) = (x^2 - xy + y^2).$$

* This is, in substance, the same as the result of A. S. Werebrusow, *L'Int. des. Sc. Math.*, 12 (1905), 268. *Vide* Dickson's *History* Vol. 2, p. 591. He gives $\quad x_1^3 + y_1^3 = x_2^3 + y_2^3 = x^3 + y^3,$

where $\quad x = \frac{x_1 y_2^2 - x_2^2 y_1}{x_2 y_2 - x_1 y_1}$ and $y = \frac{x_1^2 x_2 - y_2^2 y_1}{x_2 y_2 - x_1 y_1},$

or $\quad x = \frac{x_1^2 x_2 - y_2^2 y_1}{x_2 y_2 - x_1 y_1}$ and $y = \frac{y_1^2 y_2 - x_2^2 x_1}{x_2 y_2 - x_1 y_1}.$

$$\text{Now, } (z-xt)(z-yt)=\frac{(zy-wx)(zx-wy)}{(x+y)^2}$$

$$=\frac{-zw(x^2+y^2)+xy(z^2+w^2)}{(x+y)^2}$$

$$=-\frac{(z^2-zw+w^2)(zwt-xy)}{(x+y)^2}$$

$$=-\frac{(z^2-zw+w^2)[zw(z+w)-xy(x+y)]}{(x+y)^3}$$

$$=-\frac{(z^2-zw+w^2)[(z+w)^3-(x+y)^3]}{3(x+y)^3}$$

$$=-(z^2-zw+w^2)(t^3-1)/3.$$

Thus, when one of $(z-xt)$ or $(z-yt)$ is divisible by $(t^3-1)/3$, we obtain a second equality in integers.

If $t=l^2+3m^2$, from (6), we have

$$x=a^2(3m+l-t^2)-6ab(l-m)-3b^2(3m+l+t^2)$$
$$y=a^2(3m-l+t^2)-6ab(l+m)-3b^2(3m-l-t^2)$$
$$z=a^2(3m+l.t-1)-6ab(l-m)t-3b^2(3m+l.t+1)$$
$$w=a^2(3m-l.t+1)-6ab(l+m)t-3b^2(3m-l.t-1).$$

Now, $$\frac{z-xt}{t^3-1}=a^2+3b^2.$$

Hence $y_1=3t^2(a^2+3b^2)+(x-y)$

$$\left.\begin{aligned} \text{or, } \quad y_1&=a^2(t^2+2l)+12abm-3b^2(2l-t^2) \\ \text{and } \quad w_1&=a^2(1+2lt)+12abmt-3b^2(2lt-1). \end{aligned}\right\} \qquad (22)$$

Now, consider Type $3k^2$. Here, $l=0$ and $t=3m^2$.

$$\left.\begin{aligned}
\text{Hence } x&=-9m^4(a^2+3b^2)+3m(a^2+2ab-3b^2)\\
y&=9m^4(a^2+3b^2)+3m(a^2-2ab-3b^2)\\
z&=-(a^2+3b^2)+9m^3(a^2+2ab-3b^2)\\
w&=(a^2+3b^2)+9m^3(a^2-2ab-3b^2)\\
y_1&=9m^4(a^2+3b^2)+12abm\\
w_1&=(a^2+3b^2)+36abm^3\\
\lambda&=x+y=6m(a^2-3b^2)\\
\mu&=x-y=-18m^4(a^2+3b^2)+12mab\\
\lambda_1&=x+y_1=3m(a^2+6ab-3b^2)\\
\mu_1&=x-y_1=-18m^4(a^2+3b^2)+3m(a^2-2ab-3b^2).
\end{aligned}\right\} \qquad (23)$$

Next, consider Type 7. Here $l=2$, $m=1$, $t\equiv 1 \pmod 3$, and $(2lt+1)$ and $(l-t^2)$ are both $\equiv 0 \pmod 3$ where l may be positive or negative. Then x, y, z, w, y_1, w_1, are all multiples of 3; let $l=-2$, $m=1$. Then $2lt+1=-27$ and w_1 becomes, after division by 3,

$$-9a^2+28ab+29b^2, \text{ i.e. } (-9, 14, 29).$$

This belongs to the principal class and becomes unity for $a=5570$, $b=1417$. Dividing out by 3, we have

$$\begin{aligned} x &= (-16, 3, -50) \\ y &= (18, 1, 44) \\ z &= (2, 21, -8) \\ w &= (12, 7, -34) \\ y_1 &= (15, 2, 53) \\ w_1 &= (-9, 14, 29), \end{aligned}$$

so that

$$\left.\begin{aligned} (2, 21, -8)^3-(-16, 3, -50)^3 &= (18, 1, 44)^3-(12, 7, -34)^3 \\ &= (15, 2, 53)^3-(-9, 14, 29)^3. \end{aligned}\right\} \quad (24)$$

In this, put $a=5570$, $b=1417$. We get

$$\left.\begin{aligned} (662580696)^3-(414528234)^3 &= (549436710)^3+(377479668)^3 \\ &= (603362377)^3-1. \end{aligned}\right\} \quad (25)$$

Compare this with (19.8).

This double equality does not hold in any of the three identities associated with Type 4, viz. (10.1) and (10.2-a, b); hence $(z-xt)$ is not divisible by $(t^3-1)/3$, and w_1 which takes the form $(3, 8, -7)$ after division by 3, does not belong to the principal class. But, in Types 13 and 19, similar double equalities arise and these will be shown in the next part.

WARING'S PROBLEM FOR POWERS OF PRIMES II*

By HANSRAJ GUPTA, *Government College, Hoshiarpur.*

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1. In a recent paper[1], Pillai has examined one of my conjectures and proved the

THEOREM. *If* $l \leqslant r \leqslant 2^n - 2l - 2$ *and* $n \geqslant 20$, *then every integer not exceeding* N *is the sum of at most* I *nth powers of primes* $\geqslant 1$,

where $$N = e^{(1.6)^n/6} . 31^n,$$

$$3^n = l . 2^n + r,$$

and $$I = 2^n + l - 2.$$

He has not stated how far his theorem holds for values of $n < 20$. The object of the present paper is to study the validity of Pillai's theorem when $n \leqslant 19$, and strengthen the following

CONJECTURE. *If* $n \geqslant 7$, *then every integer can be expressed as the sum of at most* I *nth powers of primes* $\geqslant 1$.

For values of $n < 7$, this conjecture is easily shown to be false. We use the symbol $W(i, j)$ to denote the weightiest integer between i and j and $M(i, j)$ for its minimum weight. We write $a_1, a_2, a_3, a_4, \ldots,$ for $2^n, 3^n, 5^n, 7^n, \ldots$; and make a free use of the methods and ideas of the earlier paper[2].

2. Let $a_2 = q_1 a_1 - t_1$, where $t_1 + r = a_1$, and $q_1 = l + 1$.

Suppose $t_1 \geqslant q_1$, also let

$$i a_2 = q_i a_1 - t_i, \quad 1 \leqslant t_i \leqslant a_1 - 1;$$

and $c_i = t_i + i - q_i > 0$ for each value of $i \leqslant j - 1$, but not for $i = j$. Then every integer not exceeding $j a_2$ can be expressed as the sum of at most I nth powers of primes.

PROOF. Let $s_1, s_2, s_3, \ldots, s_i$ be the numbers $t_1, t_2, t_3, \ldots, t_i$ arranged in ascending order of magnitude and $d_1, d_2, d_3, \ldots, d_i$ the numbers $c_1, c_2, c_3, \ldots, c_i$ arranged in the same manner; $i < j$.

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Of the numbers: $s_1, s_2-d_1, s_3-d_2, \ldots, s_i-d_{i-1}, a_1-d_i,$
let $s_k-d_{k-1}=h_i$ be the greatest.
Then $W(i, i+1)a_2 = ia_2+A_1a_1+(s_k-1)$, where $A_1=l$ or $l-1$ according as s_k-1 is or is not less than r. Hence

$$M(i, i+1)a_2 = i+A_1+h_i-1, \\ < l;$$

because $h_i = s_k-d_{k-1} < \{a_1-(i+1-k)\}-(k-1) = a_1-i.$

3. The following two cases deserve special consideration:

(i) When $1 < r < 2^{n-1}$,

let
$$a_2 = qa_1-t,$$
$$2a_2 = (q+l)a_1-(t-r) = (2l+1)a_1-(t-r),$$
$$\ldots$$
$$ia_2 = (il+1)a_1-(t-ir+r) = (il+1)a_1-(2^n-ir),$$
$$\ldots$$

then
$$c_i = 2^n-ir+i-il-1 = (2^n-1)-i(r+l-1).$$

Now $c_i > 0$, for each value of $i \leqslant \left[\frac{2^n-1}{r+l-1}\right]$. Take

$$j = \left[\frac{2^n-1}{r+l-1}\right]+1.$$

Of the numbers mentioned in the preceding section, either

2^n-ir or $i(l-1)+r+1$ is the greatest.

Hence $h_i = \text{Max}\,(2^n-ir, il+r+1-i), i < j.$

Therefore $M(i, i+1)a_2 = i+A_1-1+\text{Max}\,(2^n-ir, il+r+1-i).$

In particular, either

$$M(j-1, j)a_2 = j-2+A_1+2^n-(j-1)r,$$
$$\leqslant 2^n+l-2-\{(j-1)(r-1)-1\} < l, \text{ since } j \geqslant 2, r > 1,$$

or
$$= j-2+A_1+(j-1)l+r+1-j+1,$$
$$= jl+r-1, \text{ since } A_1 = l-1,$$
$$= j(l+r-1)-(j-1)(r-1)$$
$$\leqslant 2^n+l+r-2-(j-1)(r-1) < l, \text{ since } j \geqslant 3, r > 1.$$

If $2^a l < m \leqslant 2^{n-a}$, where $m = l+r-1$, then

$$M(j-1, j)a_2 \leqslant (j+1)m-j(r-1), \ j \geqslant 2.$$
$$= jl+m,$$

$$\leqslant \frac{(2^n-1)l}{m}+m+l,$$

$$\leqslant 2^{n-\alpha}+2^{\alpha}l+l.$$

This result is a generalisation of one of Pillai's lemmas.

(ii) When $\quad 2^n > r > 2^{n-1},\ t-l > 1,$

let

$$a_2 = qa_1-t,$$
$$2a_2 = 2qa_1-2t,$$
$$\cdots$$
$$ia_2 = iqa_1-it,$$
$$\cdots$$

then

$$c_i = ic_1 = i(t-l).$$

Moreover

$$i \leqslant \left[\frac{2^n-1}{t}\right].$$

Take

$$j = \left[\frac{2^n-1}{t}\right]+1.$$

Of the numbers mentioned in **2**, either

2^n-ic_1 or $it-(i-1)c_1$ is the greatest.

Hence $\quad h_i = \text{Max}\ (2^n-ic_1,\ it-ic_1+c_1),\ i < j.$

Therefore

$$M(j-1,\ j)a_2 = j-2+A_1+2^n-(j-1)c_1,$$
$$= 2^n+l-2-(j-1)(c_1-1),\ \text{since}\ A_1 = l-1,$$
$$< I,\ \text{since}\ j > 2\ \text{and}\ c_1 > 1,$$

or

$$= j-2+A_1+(j-1)t-(j-2)c_1,$$
$$\leqslant 2^n+l-2-\{(j-2)(c_1-1)-1\},$$
$$< I,\ \text{since}\ j > 2\ \text{and}\ c_1 > 1.$$

Suppose $2^{\beta}(l+1) < t < 2^{n-\beta}$. Then if $h_{j-1} = 2^n-(j-1)c_1$,

$$M(j-1,\ j)a_2 \leqslant jt+l-1-(j-1)(c_1-1)$$
$$=(j-1)(l+1)+t+l-1,$$
$$\leqslant\left\{\left[\frac{(2^n-1)(l+1)}{t}\right]+t\right\}+l-1,$$
$$\leqslant 2^{n-\beta}+2^{\beta}(l+1)+l-1;$$

and if $\quad h_{j-1} = (j-1)t-(j-2)c_1,$

$$M(j-1,\ j)a_2 = (j-1)(t-c_1+1)+c_1+l-1,$$
$$=(j-1)(l+1)+t-1$$
$$\leqslant 2^{n-\beta}+2^{\beta}(l+1)-1.$$

In either case,

$$M(j-1, j)a_2 \leqslant 2^{n-\beta}+2^{\beta}(l+1)+l-1.$$

4. We proceed to consider in some detail the conjecture of **1.**

(i) $n=7$.

We have
$$a_1 = 128,$$
$$a_2 = 2187,$$
$$a_3 = 78125,$$
$$a_4 = 823543 = 10a_3+19a_2+5a_1+100,$$
$$a_5 = 19487171 = 23a_4+6a_3+35a_2+3a_1+3$$
and
$$l = 143.$$
Now
$$a_2 = 17a_1+11 = 18a_1-117,$$
$$2a_2 = 35a_1-106,$$
$$3a_2 = 52a_1-95,$$
and
$$4a_2 = 69a_1-84.$$
These equations lead to the following table of sub-intervals:

A_2	A_0	Δ
4	84	19
3	95	46
2	106	73
1	117	100.

We find that $W(a_2, a_3) = a_2+16a_1+116$ or $34a_2+16a_1+83$;

and $M(a_2, a_3) = 133.$

Again
$$a_3 = 35a_2+12a_1+44,$$
$$= 35a_2+13a_1-84,$$
$$= 36a_2-4a_1-95,$$
and
$$2a_3 = 72a_2-9a_1-62.$$
Hence we obtain the following further table of sub-intervals:

A_3	A_1	A_0	Δ
2	9	62	1
1	..	73	10
1	..	84	37
1	4	95	64.

Considering each of the sub-intervals separately, we get
$$W(a_3, a_4) = 9a_3+34a_2+16a_1+83;$$
and
$$M(a_3, a_4) = 132.$$
Proceeding in the same manner, we finally get

$$W(a_4, a_5) = a_4 + a_9 + 10a_1 + 111,$$

and
$$M(a_4, a_5) = 119.$$

It is now easily proved that every integer not exceeding 41^7 is a sum of at most 143 seventh powers of primes.

(ii) $n = 8$.

We have $a_1 = 256,$

$a_2 = 6561,$

$a_3 = 390625,$

$a_4 = 5764801 = 14a_3 + 45a_2 + 3a_1 + 38,$

$a_5 = 214358881 = 37a_4 + 2a_3 + 42a_2 + 17a_1 + 80;$

and $I = 279.$

Now $a_2 = 25a_1 + 161 = 26a_1 - 95,$

$2a_2 = 52a_1 - 190.$

Again $a_3 = 59a_2 + 14a_1 - 58,$

$= 60a_2 - 11a_1 - 219;$

$2a_3 = 120a_2 - 23a_1 - 182,$

and $3a_3 = 179a_2 - 9a_1 - 240.$

Hence we get the following table of sub-intervals:

A_3	A_2	A_1	A_0	Δ
..	1	..	95	70
1	..	..	153	56
2	..	23	182	87
..	2	..	190	140
1	..	11	219	171
3	..	9	240	73
1	..	..	248	126.

Hence $W(a_2, 2a_2) = a_2 + 24a_1 + 255;\quad M(a_2, 2a_2) = 210$

$W(2a_2, a_3) = 58a_2 + 24a_1 + 189;\ M(2a_2, a_3) = 201$

$W(a_3, a_4) = 14a_3 + 8a_1 + 247;\quad M(a_3, a_4) = 213.$

In the same manner,

$$W(a_4, a_5) = 36a_4 + 14a_3 + a_2 + 21a_1 + 214;\quad M(a_4, a_5) = 214.$$

Thus $M(a_2, a_5) = 214.$

(iii) $n = 9$.

In this case $I = 548.$

We have the following equations:

$$a_2 = 38a_1 + 227 = 39a_1 - 285.$$

$$a_3 = 99a_2 + 9a_1 - 100$$

$$= 100a_2 - 29a_1 - 327$$

$$2a_3 = 199a_2 - 20a_1 - 427$$

$$a_4 = 20a_3 + 65a_2 + 23a_1 - 64$$

$$= 20a_3 + 66a_2 - 15a_1 - 291$$

$$= 21a_3 - 33a_2 - 24a_1 - 191$$

$$= 21a_3 - 34a_2 + 14a_1 + 36$$

$$= 21a_3 - 34a_2 + 15a_1 - 476.$$

These lead to the following consolidated table of sub-intervals:

A_4	A_3	A_2	A_1	A_0	Δ
1	..	33	24	191	228
1	..	35	..	249	211
..	..	1	..	285	247
1	..	..	15	291	221
..	1	..	29	327	257
1	..	..	..	349	204
..	1	..	..	385	240
1	..	..	6	391	214
..	2	..	20	427	250
1	..	34	..	476	475
1	..	..	..	491	207.

Hence $W(a_2, a_3) = 98a_2 + 37a_1 + 284$, $M(a_2, a_3) = 419.$

$W(a_3, a_4) = 19a_3 + 98a_2 + 37a_1 + 284$, $M(a_3, a_4) = 438.$

and $W(a_4, 2a_4) = a_4 + 19a_3 + 98a_2 + 23a_1 + 248$, $M(a_4, 2a_4) = 389.$

Thus $M(a_4, a_5) \leqslant 446$, or $M(a_2, a_5) \leqslant 446.$

(iv) $n = 10$, $I = 1079$.

We have $a_2 = 57a_1 + 681 = 58a_1 - 343;$

$$2a_2 = 116a_1 - 686,$$

and $a_3 = 16$[illegible]$a_2 + 22a_1 + 12.$

We at once obtain

$$W(2a_2, a_3) = 164a_2 + 56a_1 + 1023, \quad M(2a_2, a_3) = 671.$$

(v) $n=11$, $I=2132$.

We have $a_2 = 86a_1+1019 = 87a_1-1029$,

and $a_3 \doteq 275a_2+55a_1+60$.

Therefore $W(a_2, a_3) = 274a_2+85a_1+2047$, $M(a_2, a_3) = 1463$.

(vi) $n=13$, $I=8384$.

We have the following equations:

$$a_2 = 195a_1-3117$$
$$2a_2 = 390a_1-6234$$
$$3a_2 = 584a_1-1159$$
$$4a_2 = 779a_1-4276$$
$$5a_2 = 974a_1-7393$$
$$6a_2 = 1168a_1-2318$$
$$7a_2 = 1363a_1-5435.$$

These lead to the following table of sub-intervals:

A_2	A_0	Δ
3	1159	578
6	2318	1156
1	3117	2923
4	4276	3501
7	5435	4079
2	6234	5846
5	7393	6424.

Therefore $W(7a_2, 8a\ ) = 7a_2+193a_1+6233$,

and $M(7a_2, 8a_2) = 2354$.

Since $a_3 = 765a_2+127a_1+5646$,

we get $M(7c_2, a_3) = 3111$.

(vii) $n=14$, $I=16674$.

Here $a_2 = 292a_1-1159$, $j = 1+\left[\dfrac{16383}{1159}\right] = 15$, $c_1 = 868$.

Therefore $M(14a_2, 15a_2) = 15-2+290+14.1159--13.868$
$= 5245$.

Since $a_3 = 1276a_2+27a_1+4813$,

we have $M(14a_2, a_3) = 6506$.

(viii) $n=15$, $I=33203$.

We have $a_2 = 438a_1-3477$, $j=10$, $c_1 = 3040$.

Hence $M(9a_2, 10a_2) = 4377$.

Since $a_3 = 2126a_2+360a_1+5363$,

we have $M(9a_2, a_3) = 6493$.

(ix) $n=16$; $I=66190$.

We have $a_2 = 657a_1-10431$, $j=7$, $c_1 = 9775$.

Hence $M(6, 7)a_2 = 14371.$

Since $a_3 = 3544a_2 + 462a_1 + 33769,$

we have $M(6a_2, a_3) = 17908.$

(x) $n = 17$, $I = 132055$.

We make use of the following equations and table of sub-intervals:

$$a_1 = 131072,$$
$$a_2 = 986a_1 - 96829,$$
$$2a_2 = 1971a_1 - 62586,$$
$$3a_2 = 2956a_1 - 28343,$$
$$4a_2 = 3942a_1 - 125172,$$
$$5a_2 = 4927a_1 - 90929,$$
$$6a_2 = 5912a_1 - 56686,$$
$$7a_2 = 6897a_1 - 22443,$$
$$8a_2 = 7883a_1 - 119272,$$
$$9a_2 = 8868a_1 - 85029,$$
$$10a_2 = 9853a_1 - 50786,$$
$$11a_2 = 10838a_1 - 16543,$$
$$12a_2 = 11824a_1 - 113372,$$
$$13a_2 = 12809a_1 - 79129,$$
$$14a_2 = 13794a_1 - 44886.$$

A_2	A_0	Δ
11	16543	5716
7	22443	15553
3	28343	25390
14	44886	31106
10	50786	40943
6	56686	50780
2	62586	60617
13	79129	66333
9	85029	76170
5	90929	86007
1	96829	95844
12	113372	101560
8	119272	111397
4	125172	121234.

We have $h_{14} = 19680.$

Therefore $W(14, 15)a_2 = 14a_2 + 984a_1 + 50785;$

and $M(14, 15)a_2 = 14 + 984 + 19679 = 20677.$

Finally $M(14a_2, a_3) = 26569$, because $a_3 = 5907a_2 + 827a_1 + 113740$.

(xi) $n = 18$, $I = 263619$.

We have

$$a_1 = 262144$$
$$a_2 = 1478a_1 - 28343$$
$$\cdots$$
$$9a_2 = 13302a_1 - 255087$$
$$10a_2 = 14779a_1 - 21286$$
$$\cdots$$
$$18a_2 = 26603a_1 - 248030.$$

The following is the table of sub-intervals:

A_2	A_0	Δ
10	21286	6517
1	28343	26866
11	49629	33383
2	56686	53732
...	...	...
18	248030	221445
9	255087	241794.

Hence $M(18, 19)a_2 = 35135$, because $h_{18} = 33642.$

Since $$a_3 = 9846a_2 + 591a_1 + 203827,$$
we have $$M(18a_2, a_3) = 44962.$$

(xii) $n = 19$, $I = 526502$.

We have
$$a_1 = 524288,$$
$$a_2 = 2217a_1 - 85029$$
$$\ldots$$
$$6a_2 = 13302a_1 - 510174$$
$$7a_2 = 15518a_1 - 70915$$
$$\ldots$$
$$12a_2 = 26603a_1 - 496060$$
$$13a_2 = 28819a_1 - 56801$$
$$\ldots$$
$$18a_2 = 39904a_1 - 481946$$
$$19a_2 = 42120a_1 - 42687$$
$$\ldots$$
$$24a_2 = 53205a_1 - 467832.$$

The following is the table of sub-intervals:

A_2	A_0	Δ
19	42687	586
13	56801	27995
7	70915	55404
1	85029	82813
20	127716	83399
14	141830	110808
8	155944	138217
..	..	..
24	467832	414651
18	481946	442060
12	496060	469469
6	510174	496878

$$h_{24} = 67295.$$

Hence $$M(24, 25)a_2 = 69533.$$

Since $$a_3 = 16410a_2 + 1479a_1 + 232703,$$
we have $$M(24a_2, a_3) = 85918.$$

It can now be readily shown that Pillai's theorem holds for $n > 7$. It probably holds also for $n = 7$.

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2. H. Gupta: Waring's Problem for Powers of Primes, *Journ. Ind. Math Soc.* (2) 3, (1938), 136-45. The case n=12 is considered in detail in this paper.

THE CONCEPT OF ISOTROPY IN GENERALIZED PATH-SPACES

By D. D. KOSAMBI, *Poona.*

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1. This note attempts a generalization of the concept of isotropy for spaces defined by "paths", i.e. the solution curves of

$$\ddot{x}^i + \alpha^i(x, \dot{x}, t) = 0; \; i = 1, .., n. \quad \dot{x}^i = dx^i/dt \text{ etc.} \tag{I}$$

Besides the tensor summation convention for repeated indices, we shall use the notations

$$A^{\cdot}_{\cdot,i} = \frac{\partial A^{\cdot}}{\partial x^i}; \; A^{\cdot}_{\cdot;i} = \frac{\partial A^{\cdot}}{\partial \dot{x}^i}; \; \frac{dA}{dt} = -\alpha^r A_{;r} + \dot{x}^r A_{,r} + \frac{\partial A}{\partial t}.$$

The fundamental tensorial operations for the path-space are now:

$$T^{\cdot}_{\cdot;i}; \quad \mathbf{D}T^{\cdot}_{\cdot} = \frac{dT^{\cdot}_{\cdot}}{dt} + \gamma^i_r T^r_{\cdot} - \gamma^r_j T^{\cdot}_{r}$$

$$T^{\cdot}_{\cdot}/_i = T^{\cdot}_{\cdot,i} - \gamma^r_i T^{\cdot}_{\cdot;r} + \gamma^i_{ri} T^r_{\cdot} - \gamma^r_{ik} T^{\cdot}_{r} \ldots, \tag{1.1}$$

where $\gamma^i_j = \frac{1}{2}\alpha^i_{;j}$; $\gamma^i_{jk} = \frac{1}{2}\alpha^i_{;j;k}$; $\gamma^i_{jkl} = \frac{1}{2}\alpha^i_{;j;k;l}$.

The fundamental differential invariants, besides $\dot{x}^i$ and γ^i_{jkl} are

$$\varepsilon^i = -\mathbf{D}\dot{x}^i = \alpha^i - \gamma^i_r \dot{x}^r; \quad \frac{\partial \alpha^i}{\partial t};$$

$$P^i_j = -\alpha^i_{,j} + \frac{d}{dt}\gamma^i_j + \gamma^i_r \gamma^r_j; \quad R^i_{jk} = \tfrac{1}{3}(P^i_{j;k} - P^i_{k;j}), \tag{1.2}$$

where $R^i_{jk;l}$ corresponds to the usual mixed Riemann-Christoffel tensor provided the suffixes are taken in the proper order.

By a metric for the space defined by (I), we shall mean the integrand f of any regular problem of the calculus of variations whose extremals are defined by (I). For the existence of such a metric, a set of necessary and sufficient conditions are[1]:

$$\mathbf{D}f_{;i;j} = 0; \; f_{;i;r}P^r_j - f_{;r;j}P^r_i = 0; \; |f_{;i;j}| \neq 0. \tag{1.3}$$

If, however, the space admits a metric such that any non-trivial function thereof $\phi(f)$ is also a metric, the conditions above reduce to

$$\mathbf{D}f = 0; \; f/_i = 0; \text{ along with } |f_{;i;j}| \neq 0. \tag{1.4}$$

In particular, the Finsler metric and all homogeneous metrics come under this special case. If the α^i are homogeneous of degree two, i.e. ε^i vanishes, we have $\mathrm{D}f \equiv \dot{x}^r f/_r + \partial f/\partial t$. Hence, in view of (1.4), the *metric cannot contain the parameter explicitly, and therefore, the α^i also cannot contain t explicitly, if $\varepsilon^i \doteq 0$.*

The equations of variation of (I) are most important and can be written in the invariant form:

$$\mathrm{D}^2 u^i = P^i_r u^r. \tag{II}$$

It is to be noted that the covariant form of these equations can be written[1] as

$$\mathrm{D}^2 u_i = P^r_i u_r, \tag{1.5}$$

without any device for raising or lowering the indices, merely by seeking the integrating factor for (II). As u^i is a contravariant vector u_i will be covariant; but there is no other connection between the two forms unless a metric exists.

2. The equations (II) and (1.5) are the sole means of exploring our path-space, the paths themselves being regarded, *im Kleinen* as the fundamental entities of the space. The usual mechanism for Riemannian spaces which leads to the theorem of Schur is missing. But the equations (I) always admit a dynamical interpretation, and for isotropy it would be necessary that every direction transverse to any path (trajectory) should be a principal direction. One should expect the equations (1.5) to reduce to a canonical form $\mathrm{D}^2 u_i = \phi u_i$ provided $\dot{x}^r u_r = 0$. This, however, is a restriction on P^i_j, which must, according to well-known results in tensor algebra* have the structure $\phi\delta^i_j - \dot{x}^i q_j$. All this motivates our first

DEFINITION 1. *A path-space will be called quasi-isotropic if* $P^i_j = \phi\delta^i_j - \dot{x}^i q_j$.

The implications of this definition for Riemann spaces and for those with a symmetric affine connection can be summed up in

THEOREM I. *For $n > 2$, a Riemann space is quasi-isotropic if and only if it is isotropic in the ordinary sense. A space with a symmetric affine connection is quasi-isotropic if and only if its Weyl tensor vanishes.*

For $n = 2$, nothing particular can be said. For the rest, I shall prove only the second part of the theorem, as it includes

* cf. Schouten, *Ricci-Kalkül* p. 59, ex. 5.

the first. We have $\alpha^i = \Gamma^i_{jk}\dot{x}^j\dot{x}^k$, and $P^i_j = B^i_{jkl}\dot{x}^k\dot{x}^l$ (using the notation of Eisenhart's *Non-Riemannian Geometry*). For quasi-isotropy, we must have

$$B^i_{jkl}\dot{x}^k\dot{x}^l = \phi\delta^i_j - \dot{x}^i q_j,\ \text{i. e. } \phi = a_{ij}\dot{x}^i\dot{x}^j,\ q_j = q_{jr}\dot{x}^r, \tag{2.1}$$

where $a_{ij} = a_{ji}$ and q_{ij} are functions of x alone. As $B^i_{jkl}\dot{x}^j\dot{x}^k\dot{x}^l = 0$ identically, we have $a_{ij} = \frac{1}{2}(q_{ij}+q_{ji})$, and the result follows by differentiation with respect to $\dot{x}$,

$$B^i_{jkl} = \delta^i_l a_{kj} - \delta^i_k a_{jl} + \tfrac{1}{3}(\delta^i_l \pi_{kj} + \delta^i_k \pi_{jl} + 2\delta^i_j \pi_{kl}),$$

where $$\pi_{ij} = \tfrac{1}{2}(q_{ij} - q_{ji}). \tag{2.2}$$

But the projective curvature tensor for a space with symmetric affine connection is given by

$$W^i_{jkl} = B^i_{jkl} + \frac{2}{n+1}\,\delta^i_j\beta_{kl} + \frac{1}{n-1}\left\{\delta^i_k B_{jl} - \delta^i_l B_{jk}\right\} + \frac{2}{n^2-1}\left\{\delta^i_l\beta_{jk} - \delta^i_k\beta_{jl}\right\}. \tag{2.3}$$

where $$B_{jk} = B^i_{jki}\quad,\quad \beta_{jk} = \tfrac{1}{2}(B_{jk} - B_{kj}).$$

By direct substitution from (2.2), it follows here that

$$W^i_{jkl} = 0.$$

For the converse, we see that for $W^i_{jkl} = 0$, the tensor P^i_j is given by

$$B^i_{lkj}\dot{x}^k\dot{x}^l = \frac{1}{n-1}\delta^i_j B_{lk}\dot{x}^k\dot{x}^l - \dot{x}^i\left\{\frac{2}{n+1}\beta_{kj} + \frac{1}{n-1}B_{kj} - \frac{2}{n^2-1}\beta_{kj}\right\}\dot{x}^k, \tag{2.4}$$

which is precisely of the form required by our definition. The proposed definition, therefore, covers the classical Riemannian isotropy, and its generalization, the projectively flat space of symmetric affine connection for $n > 2$.

3. To deal with the general α^i, we shall require a certain number of identities, in particular,

$$\dot{x}^i_{/j} = -\varepsilon^i_{;j}\ ;\ \mathbf{D}\,\varepsilon^i = -P^i_r\dot{x}^r + \partial\alpha^i/\partial t. \tag{3.1}$$

Then, the analogue of the Bianchi identities

$$R^i_{kj/l} + R^i_{kl/j} + R^i_{lj/k} = 0. \tag{3.2}$$

To clarify the relation of this with the usual form of the identity, we can restate it in the 4-index tensor:

$$R^i_{jk;m/l} + R^i_{kl;m/j} + R^i_{lj;m/k} = R^r_{kj}\gamma^i_{rml} + R^r_{lk}\gamma^i_{rmj} + R^r_{jl}\gamma^i_{rmk}. \tag{3.3}$$

These identities hold for all sufficiently differentiable α^i, and each of (3.2) and (3.3) can be derived from the other.

For quasi-isotropy, with $P^i_j = \phi\delta^i_j - \dot{x}^i q_j$, we have, as a consequence of (3.2),

$$\delta^i_j \{ \phi_{;k/l} - \phi_{;l/k} + q_{k/l} - q_{l/k} \} + \delta^i_k \{ \phi_{;l/j} - \phi_{;j/l} + q_{l/j} - q_{j/l} \}$$
$$+ \delta^i_l \{ \phi_{;j/k} - \phi_{;k/j} + q_{j/k} - q_{k/j} \}$$
$$-2\{ (\dot{x}^i \pi_{jk})_{/l} + (\dot{x}^i \pi_{kl})_{/j} + \dot{x}^i \pi_{lj})_{/k} \} = 0, \qquad (3.4)$$

where
$$2\pi_{ij} = q_{i;j} - q_{j;i}.$$

As expected, this vanishes identically for $n = 2$. For $n = 3$, it reduces to

$$\phi_{;k/l} - \phi_{;l/k} + q_{k/l} - q_{l/k} = 2\{ (\dot{x}^j \pi_{jk})_{/l} + (\dot{x}^j \pi_{kl})_{/j} + (\dot{x}^j \pi_{lj})_{/k} \}. \quad (3.5)$$

The odd appearance of this expression is accounted for by the fact that for $n = 3$, and for no other case, it is possible to display an anti-symmetric two-index covariant tensor as a contravariant vector. In (3.5), it is immaterial whether the repeated index i is summed or not, as (3.4) is identically satisfied if any two subscripts are the same, and the superscript i must therefore be one of j, k, l, which can be taken as distinct among themselves. From this point of view, (3.5) says that the curl-vector on the left is twice the divergence of the contravariant tensor formed by the product of $\dot{x}^i$ with the bi-vector π_{kl}. We shall not follow this any further.

For $n \geqslant 4$, (3.3) can be broken up by first taking $i \neq j \neq k \neq l$ which implies the vanishing of the last bracket by itself for those indices. Then take, say, $i = j \neq k \neq l$, and finally, sum by contracting i and j. A comparison of the various results shows that we must have

$$(a) \quad \phi_{;k/l} - \phi_{;l/k} + q_{k/l} - q_{l/k} = 0;$$
$$(b) \quad (\dot{x}^i \pi_{jk})_{/l} + (\dot{x}^i \pi_{kl})_{/j} + (\dot{x}^i \pi_{jl})_{/k} = 0. \qquad (3.6)$$

Of these, the second has further consequences obtained by writing it in the form—by making use of (3.1)

$$(b) \quad \dot{x}^i (\pi_{jk/l} + \pi_{kl/j} + \pi_{lj/k}) = \varepsilon^i_{;l}\pi_{jk} + \varepsilon^i_{;j}\pi_{kl} + \varepsilon^i_{;k}\pi_{lj}. \qquad (3.61)$$

Hence, either, each side of this vanishes separately, or $\dot{x}^i$ is a factor of each side. This would mean $\varepsilon^i_{;j} = \dot{x}^i \sigma_j$. Differentiating, we get $\varepsilon^i_{;j;k} = \delta^i_k \sigma_j + \dot{x}^i \sigma_{j;k}$. By alternating these and subtracting, we have $\dot{x}^i (\sigma_{k;j} - \sigma_{j;k}) = \delta^i_k \sigma_j - \delta^i_j \sigma_k$. For $i \neq j \neq k$, we have $\sigma_{k;j} - \sigma_{j;k} = 0$, hence $\delta^i_k \sigma_j - \delta^i_j \sigma_k = 0$ for all i, j, k and by contracting

i, j in the last step, we see that $\sigma_j = 0$. This means $\varepsilon^i = v^i(x, t)$, $\alpha^i = \alpha^{*i} + v^i$, where α^{*i} is homogeneous of degree two in $\dot{x}$. Moreover, in any case, each side of (2.6) vanishes separately.

If the right side of (3.61) vanishes, and π_{ij} is not identically zero, and for even values of n, the determinant π_{ij} does not vanish, and we can solve $\pi_{ij}\pi^{jk} = \delta_i^k$. Contracting the right side of (3.61), with π^{jk}, we get $(n-2)\,\varepsilon^i_{;l} = 0$, which is again the conclusion of the last paragraph. For odd n, the determinant of π_{ij} will vanish identically.

THEOREM II. *For quasi-isotropy, $n = 3$, the equations* (3.5) *must be satisfied. For $n \geqslant 4$, we must have*

$$\phi_{;k/l} - \phi_{;l/k} + q_{k/l} - q_{l/k} = 0$$

and

$$\pi_{jk/l} + \pi_{kl/j} + \pi_{jl/k} = \varepsilon^i_{;j}\pi_{kl} + \varepsilon^i_{;k}\pi_{lj} + \varepsilon^i_{;l}\pi_{jk} = 0.$$

I shall make one more general remark in this section about quasi-isotropic spaces. If we perform the analogue of a projective change of connection by replacing α^i with $\alpha^i + \dot{x}^i\psi$, we get

$$\bar{P}^i_j = P^i_j + \tfrac{1}{2}\delta^i_j\left\{D\psi + \tfrac{1}{4}\psi^2 - \psi\psi_{;r}\dot{x}^r\right\} - \tfrac{1}{2}\varepsilon^i\psi_{;j} + \psi\varepsilon^i_{;j}$$
$$- \dot{x}^i\left\{-\tfrac{1}{2}D\psi_{;j} + \psi_{/j} + \tfrac{1}{2}\psi\psi_{;j;r}\dot{x}^r - \tfrac{1}{4}\psi_{;j}\psi_{;r}\dot{x}^r\right\}. \qquad (3.7)$$

As in the case of projectively flat spaces with a symmetric affine connection, it is possible to make P^i_j vanish if the space is quasi-isotropic, and $\varepsilon^i = \partial\alpha^i/\partial t = 0$, but not in general.

4. We now consider quasi-isotropic spaces for which $\varepsilon^i = \partial\alpha^i/\partial t = 0$, and a metric exists. In view of (3.1), we must have $\phi = q_r\dot{x}^r$. The metric can, without loss of generality, be taken as homogeneous[2] in $\dot{x}$, and there are two cases to be distinguished: homogeneity of degree zero, and homogeneity of any other degree. For the non-zero case, the metric can always be replaced[2] by one of degree two; we could demand a metric of degree one, but then the determinant $|f_{;i;j}|$ will vanish; we have merely followed the device used in Finsler spaces of replacing the metric f by f^2. In either case, we apply the second part of (1.3), or the integrability conditions of (1.4), which are $P^r_i f_{;r} = R^r_{ij} f_{;r} = 0$. We shall have to use the condition of homogeneity $\dot{x}^r f_{;r} = 0$, or $= 2f$, respectively. The results are:

$$\text{case 1}: \; P^i_j = \mu \dot{x}^i f_{;j} \quad ; \quad \mu_{;r}\dot{x}^r = 2\mu,$$

$$\text{case 2}: \; P^i_j = \lambda(f\delta^i_j - \tfrac{1}{2}\dot{x}^i f_{;j}) \;, \; \lambda_{;r}\dot{x}^r = 0. \qquad (4.1)$$

The homogeneity restrictions on μ, λ follow from the fact that in any case, the curvature tensor P^i_j must be homogeneous of degree two with $\dot{x}^i$. To apply (3.6), we keep in mind the fact that $f_{/i} = f_{;i/j} = 0$ are consequences of (1.4). We then obtain:

$$\text{case 1}, \; \mu_{/i} f_{;j} - \mu_{/j} f_{;i} = 0,$$

$$\text{case 2}. \; 2(\lambda_{;i/j} - \lambda_{;j/i})f + 3(\lambda_{/j} f_{;i} - \lambda_{/i} f_{;j}) = 0. \qquad (4.2)$$

In each case, (3.61) is identically fulfilled, as can be verified by differentiation of (4.2) with respect to $\dot{x}^k$, and addition after cyclic rotation of the subscripts. In case 1, we can contract (4.2) with $\dot{x}^i$, to arrive at the conclusion that $\mathbf{D}\mu = 0$. The equations of variation have the form $\mathbf{D}^2 u_i = 0$, a case of neutral equilibrium. To get any further, we shall have to formulate another

DEFINITION 2. *A quasi-isotropic space will be called isotropic if the function ϕ can, by suitable choice of the parameter t, be made to have the same value for the entire sheaf of paths through each point.*

With Definition 1, this amounts to saying that the "curvature is locally constant". For case 2 of this section, we see that the function λ will, for full isotropy, have to be a function of position alone, not of direction, inasmuch as it is of degree zero in $\dot{x}$, and hence, a change of parameter will not affect it at all. Here, $\phi = \lambda f$, and, from (1.4), $\mathbf{D}f = 0$, which means that the integral $f = const.$ along the path exists, and choice of parameter will fix this constant in value. We have then, $\lambda_{;i} = 0$, and (4.2) (2) reduces to

$$\lambda_{/i} f_{;j} - \lambda_{/j} f_{;i} = 0 \;\; ; \;\; \lambda_{/i} = \lambda_{,i}. \qquad (4.3)$$

This can be possible only if $\lambda_{/i} = 0$, or $f_{;i} = \sigma\lambda_{/i}$. The latter would lead to $f_{;i;j} = \lambda_{;i}\sigma_{;j}$, and to $|f_{;ij}| = 0$, a contradiction of our hypothesis. Hence, $\lambda_{;i} = \lambda_{/j} = 0$, or λ is a constant, exactly as for the Riemannian case.

THEOREM III. *Isotropy for a space admitting a metric f homogeneous of degree two in $\dot{x}$ is possible if and only if $P^i_j = c(f\delta^i_j - \frac{1}{2}\dot{x}^i f_{;j})$ where c is an absolute constant. Unless $P^i_j = 0$, the isotropic space cannot admit two essentially distinct metrics.*

The latter part of the theorem follows from (4.1). For, if there were two distinct metrics, one not being a function of the other, there would be two distinct metrics of degree two, and their ratio, satisfying (1.4), would also be a metric as the discriminant will not vanish in general. Then, P^i_j admits a metric of degree zero, and so has *both* the forms in (4.1), and hence vanishes.

5. For the general non-homogeneous metric and general α^i, we shall have $P^i_j = \phi\delta^i_j - \lambda \dot{x}^i f_{;j;r}\dot{x}^r$. Rather than deal with this, I consider an allied question: When can a given space be "immersed" into a quasi-isotropic space by taking the parameter t as an additional cordinate x^0 with a new parameter s, so as to make the the new $\varepsilon^i = \partial\alpha^i/\partial s = 0$? The new path equations become[3]

$$\begin{aligned} x''^i + x'^{02}\left\{\alpha^i + \beta\dot{x}^i\right\} &= 0 \\ x''^0 + x'^{02}\beta &= 0 \end{aligned} \qquad \left\{ \begin{array}{l} i = 1, \ldots, n, \\ \beta(x, \dot{x}, t) \text{ arbitrary}, \\ x'^i = dx^i/ds\,;\ \dot{x}^i = dx^i/dt = x'^i/x'^0. \end{array}\right. \quad \text{(III)}$$

The new curvature tensor is given by

$$\frac{1}{x'^{02}}\bar{P}^i_j = P^i_j + \tfrac{1}{2}\delta^i_j\left\{ D\beta - \tfrac{1}{2}\beta^2 \right\} - \dot{x}^i\left\{ \beta/_j - \tfrac{1}{2}D\beta_{;j} - \tfrac{1}{4}\beta\beta_{;j} \right\};$$

$$\frac{1}{x'^{02}}\bar{P}^0_j = -\beta/_j + \tfrac{1}{2}D\beta_{;j} + \tfrac{1}{4}\beta\beta_{;j}\ ;\quad \bar{P}^0_0 = -\bar{P}^0_r\dot{x}^r;$$

$$\bar{P}^i_0 = \dot{x}^r\bar{P}^i_r\,, \qquad (5.1)$$

where the summations for repeated indices are from 1 to n. For the new space to be quasi-isotropic, $\bar{P}^0{}_j = -x'^0\bar{q}_j$ which means, from (5.1), that $q_j = 0$, in addition, of course, to quasi-isotropy for the original space.

THEOREM IV. *A path-space may be immersed in a quasi-isotropic space with* $\bar{\varepsilon}^i = \partial\bar{\alpha}^i/\partial s = 0$ *only if its original curvature tensor has the form* $P^i_j = \phi\delta^i_j$.

If the original space had a metric, the new metric (of degree 2) will be $f^2x'^{02}$, and the function of immersion[3] is given by $\beta = Df/f$. The condition for quasi-isotropy of the $n+1$ space can now be expressed according to (4.1) and (5.1) as

$$\left\{\tfrac{1}{2}D\beta - \tfrac{1}{4}\beta^2 + \phi\right\}f_{;j} - f\left\{\beta/_j - \tfrac{1}{2}D\beta_{;j} - \tfrac{1}{4}\beta\beta_{;j}\right\} = 0;\ \beta = \frac{Df}{f}. \qquad (5.2)$$

For full isotropy according to Theorem III, these can be further reduced in an obvious manner to give

$$\{ \tfrac{1}{2}D\beta-\tfrac{1}{4}\beta^2+\phi \}_{;j} = 2 \{\beta_{1j}-\tfrac{1}{2}D\beta_{;j}-\tfrac{1}{4}\beta\beta_{;j} \} \; ; \; cf^2 = \tfrac{1}{2}D\beta-\tfrac{1}{4}\beta^2+\phi,$$
$$\beta = Df/f. \tag{5.3}$$

The interest of these results lies in their applicability to the relationship between a space and a space-time, as required by modern physical theories. For example, an M_3 occurs in Milne's cosmology, defined after taking the velocity of light as unity by[4]:

$$p^i = x^i, \quad g_{ii} = 1, \; g_{ij} = 0, \; i \neq j, \quad Y = 1-g_{ij}\dot{x}^i\dot{x}^j,$$
$$X = t^2-g_{ij}p^ip^j, \quad Z = t-g_{ij}p^i\dot{x}^j, \quad \xi = Z^2/XY,$$
$$\alpha^i = (t\dot{x}^i-p^i)\frac{Y}{X}G(\xi). \tag{5.4}$$

$$X^2P^i_j = \delta^i_j \{ \tfrac{8}{2}XYG-\tfrac{8}{4}t^2Y^2G^2+tYZ[-G+G^2+(1-\xi)(1+G)G'] \}$$
$$-\Gamma(p^i-t\dot{x}^i)(Yp_j-Z\dot{x}_j)$$
$$\Gamma = -G(2+G)+(2\xi-1)(1+G)G'+\xi(\xi-1)\{ 2(1+G)G''-G'^2 \}. \tag{5.5}$$

As the "fundamental quantities" X, Y, ξ, are not to be restricted, the condition for immersibility into a quasi-isotropic space is the same as that for quasi-isotropy of the M_3 itself, i. e. $\Gamma = 0$.

The space into which this can be immersed is[4] the K_4 defined by

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1, \; g_{ij} = 0, \; i \neq j, \; p_j = g_{jr}p^r \text{ etc.}, \; p^i = x^i$$
$$\mathbf{X} = p_rp^r \quad \mathbf{Y} = \dot{x}^r\dot{x}_r \quad \mathbf{Z} = \dot{x}^rp_r \quad \xi = \mathbf{Z}^2/\mathbf{XY}$$
$$\alpha^i = -\frac{p^i\mathbf{Y}}{\mathbf{X}}G(\xi)+\frac{2\dot{x}^i\mathbf{Z}}{\mathbf{X}}\gamma(\mathbf{X}, \xi). \tag{5.6}$$

$$\mathbf{X}^2P^i_j = A\mathbf{XY}\delta^i_j+\dot{x}^i\{ (C\xi-A)\mathbf{X}\dot{x}_j-C\mathbf{Z}p_j \}-p^i\Gamma\{ \mathbf{Z}\dot{x}_j-\mathbf{Y}p_j \}$$
$$A = G\{ 1+\gamma+2\gamma_2\xi(1-\xi) \}+\gamma+2\xi(\mathbf{X}\gamma_1-\gamma)+2\gamma_2\xi(1-)\xi-\xi\gamma^2$$
$$C = G'\{ 1+\gamma+2\gamma_2\xi(1-\xi) \}+2(\mathbf{X}\gamma_1-\gamma)-\gamma^2-6\gamma_2(1-\xi+G)$$
$$-2\gamma\gamma_2\xi-4\gamma_{12}\mathbf{X}\xi+8G\gamma_2\xi-4\gamma_{22}\xi(1-\xi)(1+G)$$
$$\Gamma \text{ as in (5.5)}; \quad \gamma_1 = \partial\gamma/\partial\mathbf{X}, \quad \gamma_2 = \partial\gamma/\partial\xi \text{ etc.} \tag{5.7}$$

The condition for quasi-isotropy of K_4 is again $\Gamma = 0$. The differential equation $\Gamma = 0$ has only two constant soutions for G, i.e. $G = 0$, $G = -2$. For both of these, a metric exists in terms of the fundamental quantities: $\gamma = G = 0$, $f = \mathbf{Y}$, $\gamma = G = -2$, $f = \mathbf{Y}/\mathbf{X}^2$. Both of these are actually flat spaces, which is obvious for the first from the definition of K_4 and is seen for the second

by choosing the new coordinate system: $\bar{x}^i = x^i/\mathbf{X}$, which gives a Galilean metric again for the space, though the transformation is singular for the origin and for all light-tracks from it.

The condition for the existence of a metric homogeneous of degree zero for the K_4 has been given elsewhere[4] as

$$1+(1-\xi)(G-\xi G')+\xi\gamma = 0. \qquad (5.8)$$

In this case, the findings of (4.1) are confirmed by the fact that the coefficient of the δ^i_j term in P^i_j above becomes $\mathbf{XY}(\xi-1)\Gamma$, which vanishes for isotropy.

6. Our definition of isotropy allows one further analogue of the Riemannian case. For $n=2$, with homogeneous non-parametric α^i, and a metric homogeneous of degree zero or two we find the following results identically true:

$$P^r_1 f_{;r} = 0,\ P^r_2 f_{;r} = 0;\ P^1_r \dot{x}^r = 0,\ P^2_r \dot{x}^r = 0,\ \tfrac{1}{2} f_{;r}\dot{x}^r = kf\ (k=0, \text{ or } 1). \qquad (6.1)$$

But these can be solved by elementary algebra, and lead at once to the result:

$$P^i_j = \lambda(kf\delta^i_j - \tfrac{1}{2}\dot{x}^i f_{;j}) \text{ identically, for } n=2, \text{ and a homogeneous metric.} \qquad (6.2)$$

Hence

If $n = 2$ and a homogeneous metric exists, the "surface" is always identically quasi-isotropic.

In conclusion, I gratefully acknowledge the valuable aid given by Mr. V. Seetharaman in checking this note, particularly by his independent and direct verification of (3.2), (3.3), (5.5) and (5.7).

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S. MAHADEVA IYER, Esq., M.A., L.T.,
Presidency College, Madras.

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